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Oscillation criteria for second order forced elliptic differential equations with mixed nonlinearities

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Abstract

In the case of oscillatory potentials, we establish some oscillation criteria for the second order forced elliptic differential equation with mixed nonlinearities

$$\operatorname{div}(A(x)\nabla y) + \langle b(x), \nabla y\rangle + C(x, y) = e(x).$$

When $N = 1$, our results extend those of Kong [Q. Kong, Interval criteria for oscillation of second order linear ordinary differential equations, J. Math. Anal. Appl. 229 (1999) 258–270] for certain second order linear ordinary differential equations, Sun and Meng [Y.G. Sun, F.W. Meng, Interval criteria for oscillation of second order differential equations with mixed nonlinearities, Appl. Math. Comput. (2007), doi:10.10.1016/j.amc.2007.08.042] for certain second order differential equations with mixed nonlinearities, Yang [Q. Yang, Interval oscillation criteria for a second order nonlinear ordinary differential equations with oscillatory potential, Appl. Math. Comput. 135 (2003) 49–64] for a forced second order superlinear ordinary differential equation. When $N \geq 2$, our theorems are more general, and improve the oscillation criteria of Zhuang [R.-K. Zhuang, Annual oscillation criteria for second order nonlinear elliptic differential equations, J. Comput. Anal. Math. (2007), doi:10.1016/j.cam.2007.06.031] for a forced second order undamped elliptic differential equation.

Keywords: Oscillation; Forced elliptic differential equations; Second order; Damped; Mixed nonlinearities

1. Introduction and preliminaries

Consider the forced second order damped elliptic differential equation with mixed nonlinearities

$$\operatorname{div}(A(x)\nabla y) + \langle b(x), \nabla y\rangle + C(x, y) = e(x) \tag{1.1}$$

in $\Omega(r_0)$, where $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $|x| = [\sum_{i=1}^{N} x_i^2]^{1/2}$, $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_N)$. The operator $\langle\, ,\rangle$ denotes the usual scalar product in $\mathbb{R}^N$, and $\Omega(r_0) = \{x \in \mathbb{R}^N : |x| \geq r_0\}$ for some $r_0 \geq 0$.

Throughout this paper we shall assume that

(A1) $A(x) = (a_{ij}(x))_{N\times N}$ is a real symmetric positive definite matrix function with $a_{ij} \in \mathbf{C}^{1+\mu}_{\mathrm{loc}}(\Omega(r_0), \mathbb{R})$ for all i, j, $\mu \in (0, 1)$;

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(A2) $b_i \in \mathbf{C}^{\mu}_{\text{loc}}(\Omega(r_0), \mathbb{R})$ for all i;
(A3) $C \in \mathbf{C}^{\mu}_{\text{loc}}(\Omega(r_0) \times \mathbb{R}, \mathbb{R})$ with $C(x, -y) = -C(x, y)$ for all $(x, y) \in \Omega(r_0) \times \mathbb{R}$. Suppose that there exist functions $q, q_i \in \mathbf{C}^{\mu}_{\text{loc}}(\Omega(r_0), \mathbb{R})$, $i = 1, 2, \ldots, m$, such that

$$yC(x, y) \geq q(x)y^2 + \sum_{i=1}^{m} q_i(x)y^{\alpha_i+1} \quad \text{for } x \in \Omega(r_0),\ y > 0,$$

and $\alpha_1 > \cdots > \alpha_k > 1 > \alpha_{k+1} > \cdots > \alpha_m > 0$;
(A4) $e \in \mathbf{C}^{\mu}_{\text{loc}}(\Omega(r_0), \mathbb{R})$.

A function $y \in \mathbf{C}^{1+\mu}_{\text{loc}}(\Omega(r_0), \mathbb{R})$ with the property $a_{ij}\partial y/\partial x_i \in \mathbf{C}^{1+\mu}_{\text{loc}}(\Omega(r_0), \mathbb{R})$ for all i, j is said to be a solution of Eq. (1.1) in $\Omega(r_0)$, if $y(x)$ satisfies Eq. (1.1) for all $x \in \Omega(r_0)$. For the existence of solutions of Eq. (1.1), we refer the reader to the monograph [1]. We restrict our attention only to the nontrivial solution $y(x)$ of Eq. (1.1), i.e., $\sup\{|y(x)| : |x| > r\} > 0$ for any $r \geq r_0$. A nontrivial solution $y(x)$ of Eq. (1.1) is called oscillatory in $\Omega(r_0)$ if the set $\{x \in \Omega(r_0) : y(x) = 0\}$ is unbounded; otherwise it is said to be nonoscillatory in $\Omega(r_0)$. Eq. (1.1) is oscillatory if all of its solutions are oscillatory in $\Omega(r_0)$.

During the last decades, the oscillation problem for a special case of Eq. (1.1) such as the damped elliptic differential equation

$$\operatorname{div}(A(x)\nabla y) + \langle b(x), \nabla y\rangle + C(x, y) = 0 \tag{1.2}$$

has been studied extensively, e.g., see [2–14] and references therein.

In [15], one can find some results sufficient for the forced oscillation of another special case of Eq. (1.1)

$$\operatorname{div}(A(x)\nabla y) + q(x)|y|^{\alpha-1}y = e(x), \quad \alpha \geq 1. \tag{1.3}$$

Zhuang [15], by using the partial Riccati technique and the two-parametric averaging function $H(r, s)$, developed earlier by Noussair and Swanson [16] and Philos [17], have extended Kong's oscillation theorem [18] to Eq. (1.3).

Inspired by the ideas of Kong [18], Noussair and Swanson [16], Sun and Meng [19], Sun and Wong [20]; in this paper, we shall establish some forced oscillation criteria for Eq. (1.1) which extend the results of Kong [18] to Eq. (1.1). When $N = 1$, our results improve those of Kong [18] for certain second order linear ordinary differential equations, Sun and Meng [19] for certain second order differential equations with mixed nonlinearities, and Yang [21] for a forced second order superlinear ordinary differential equation. When $N \geq 2$, our results are more general, and improve the oscillation criteria of Zhuang [15] for Eq. (1.3). In particular, for the case examined, some interesting corollaries are established, and two examples that illustrate the conditions of our results are also included.

Before proceeding, we shall define some notation to be used in this paper.

$S(a) = \{x \in \mathbb{R}^N : |x| = a\}$,
$\Omega[a, b] = \{x \in \mathbb{R}^N : a \leq |x| \leq b\}$,
$\Omega(a, b) = \{x \in \mathbb{R}^N : a < |x| < b\}$,
$A^{-1}(x)$ denotes the inverse of $A(x)$,
$\nu(x) = x/|x|$ denotes the outside normal unit vector to the sphere $S(|x|)$,
$d\sigma$ represents the integral element of the sphere $S(|x|)$,
$\lambda_{\max}(x)$ denotes the largest eigenvalue of the matrix $A(x)$.

Note that $A(x)$ is a real symmetric positive definite matrix function (ellipticity condition), it follows that $\lambda_{\max}(x) > 0$ for all $x \in \Omega(r_0)$.

For $\rho \in \mathbf{C}^1(\Omega(r_0), \mathbb{R}^+)$ and $h_1, h_2 \in \mathbf{C}(\mathbb{R}^2, \mathbb{R})$, we define

$$\Theta_1(|x|, r) = h_1(|x|, r)\nu(x) - b(x)A^{-1}(x) + \frac{\nabla\rho(x)}{\rho(x)},$$
$$\Theta_2(r, |x|) = h_2(r, |x|)\nu(x) + b(x)A^{-1}(x) - \frac{\nabla\rho(x)}{\rho(x)}.$$

Following Philos [17], we introduce a class of functions $\mathcal{H}$. Let $D = \{(r, s) : r \geq s \geq r_0\}$, $H_1, H_2 \in \mathbf{C}(D, \mathbb{R})$. A pair of real-valued functions (H_1, H_2) is said to belong to a function set $\mathcal{H}$, denoted by $(H_1, H_2) \in \mathcal{H}$, if there exist functions $h_1, h_2 \in \mathbf{C}(D, \mathbb{R})$ satisfying the following conditions:

(C1) $H_i(r, r) = 0$ for $r \geq r_0$, and $H_i(r, s) > 0$ for $r > s \geq r_0$, $i = 1, 2$;
(C2) $H_i(r, s)$, $i = 1, 2$, has partial derivatives on D, and

$$\frac{\partial}{\partial r} H_1(r, s) = h_1(r, s) H_1(r, s), \qquad \frac{\partial}{\partial s} H_2(r, s) = h_2(r, s) H_2(r, s).$$

In order to prove our results we will need the following lemmas. The first one is due to Sun and Wong [20], the second one is the well-known arithmetic-geometric mean inequality which can found in Hardy et al. [22], and the last one is a result on the maximal value of a function, which can be proved by a direct computation.

Lemma 1.1 ([20]). *Let $\{\alpha_i\}$, $i = 1, \ldots, m$, be the $m-$ tuple satisfying $\alpha_1 > \cdots > \alpha_k > 1 > \alpha_{k+1} > \cdots > \alpha_m > 0$. Then there exists an $m-$ tuple $(\eta_1, \ldots, \eta_m)$ with $\sum_{i=1}^m \eta_i < 1$ and $0 < \eta_i < 1$ such that $\sum_{i=1}^m \alpha_i \eta_i = 1$.*

Lemma 1.2 ([22]). *It holds that*

$$\sum_{i=0}^m q_i a_i \geq \prod_{i=0}^m a_i^{q_i},$$

where $a_i \geq 0$, $q_i > 0$ with $\sum_{i=0}^m q_i = 1$.

Lemma 1.3. *Let $A \geq 0$, $B > 0$ and $x > 0$. Then*

(i) $\gamma > 1$, $Ax^\gamma + B \geq \gamma(\gamma - 1)^{1/\gamma - 1} A^{1/\gamma} B^{1-1/\gamma} x$;
(ii) $0 < \gamma < 1$, $Ax^\gamma - B \leq \gamma(1 - \gamma)^{1/\gamma - 1} A^{1/\gamma} B^{1-1/\gamma} x$.

2. Main results

In this section we formulate and prove our results.

Theorem 2.1. *Assume that for any $r \geq r_0$, there exist a_1, b_1, a_2, b_2 such that $r \leq a_1 < b_1 \leq a_2 < b_2$ and*

$$\begin{cases} q_i(x) \geq 0 & x \in \Omega[a_1, b_1] \cup \Omega[a_2, b_2],\ i = 1, \ldots, m, \\ e(x) \leq 0, & x \in \Omega[a_1, b_1];\ e(x) \geq 0,\ x \in \Omega[a_2, b_2]. \end{cases} \tag{2.1}$$

If there exist $c_i \in (a_i, b_i)$, $i = 1, 2$, $(H_1, H_2) \in \mathcal{H}$ and $\rho \in \mathbf{C}^1(\Omega[a_1, b_1] \cup \Omega[a_2, b_2], \mathbb{R}^+)$ such that

$$\begin{aligned} &\frac{1}{H_1(c_i, a_i)} \int_{\Omega[a_i, c_i]} H_1(|x|, a_i) \rho(x) \left[Q_1(x) - \frac{1}{4} \lambda_{\max}(x) |\Theta_1(|x|, a_i)|^2 \right] \mathrm{d}x \\ &\quad + \frac{1}{H_2(b_i, c_i)} \int_{\Omega[b_i, c_i]} H_2(b_i, |x|) \rho(x) \left[Q_1(x) - \frac{1}{4} \lambda_{\max}(x) |\Theta_2(b_i, |x|)|^2 \right] \mathrm{d}x > 0, \end{aligned} \tag{2.2}$$

for $i = 1, 2$, where

$$Q_1(x) = \begin{cases} \alpha_1(\alpha_1 - 1)^{1/\alpha_1 - 1} |e(x)|^{1-1/\alpha_1} q_1^{1/\alpha_1}(x) + q(x), & m = 1, \\ \left(\prod_{i=0}^m \eta_i^{-\eta_i} \right) |e(x)|^{\eta_0} \left(\prod_{i=1}^m q_i^{\eta_i}(x) \right) + q(x), & m > 1, \end{cases}$$

$\eta_1, \ldots, \eta_m$ are positive constants given in Lemma 1.1, and $0 < \eta_0 < 1$ with $\sum_{i=0}^m \eta_i = 1$, then Eq. (1.1) is oscillatory.

Proof. Suppose that $y = y(x)$ is a nonoscillatory solution of Eq. (1.1). Without loss of generality, we may assume that $y(x) > 0$ for all $x \in \Omega(a_0)$, $a_0 \geq r_0$, where a_0 depends on the solution $y(x)$. When $y(x)$ is eventually negative, the proof follows the same argument using the annulus $\Omega[a_2, b_2]$ instead of $\Omega[a_1, b_1]$. In view of (A3), (1.1) can be rearranged as

$$\operatorname{div}(A(x)\nabla y) + \langle b(x), \nabla y\rangle + q(x)y + \sum_{i=1}^{m} q_i(x)y^{\alpha_i} \le e(x). \tag{2.3}$$

By the assumption, we can choose $a_1 > b_1 \ge a_0$ such that $q_i(x) \ge 0$ for all i and $e(x) \le 0$ for $x \in \Omega[a_1, b_1]$.

Next, we consider the following two cases: (i) $m = 1$; (ii) $m > 1$.

(i) $m = 1$. In this case, (2.3) reduces to

$$\operatorname{div}(A(x)\nabla y) + \langle b(x), \nabla y\rangle + q(x)y + q_1(x)y^{\alpha_1} \le e(x). \tag{2.4}$$

By Lemma 1.3 (i), by virtue of $\alpha_1 > 1$, we have

$$q_1(x)y^{\alpha_1} + |e(x)| \ge \alpha_1(\alpha_1 - 1)^{1/\alpha_1 - 1}|e(x)|^{1-1/\alpha_1} q_1^{1/\alpha_1}(x)y.$$

Combining the above inequality with (2.4), we get

$$\operatorname{div}(A(x)\nabla y) + \langle b(x), \nabla y\rangle + Q_1(x)y \le 0. \tag{2.5}$$

(ii) $m > 1$. Let $u_0 = \eta_0^{-1}|e(x)|y^{-1}$ and $u_i = \eta_i^{-1}q_i(x)y^{\alpha_i - 1}$, where $\eta_i > 0$, $i = 1, \dots, m$, are chosen to satisfy Lemma 1.1 with $\sum_{i=0}^{m} \eta_i = 1$. Then, by Lemma 1.2, we have

$$\sum_{i=1}^{m} q_i(x)y^{\alpha_i} + |e(x)| \ge \left(\prod_{i=0}^{m} \eta_i^{-\eta_i}\right)|e(x)|^{\eta_0}\left(\prod_{i=1}^{m} q_i^{\eta_i}(x)\right)y.$$

Combining the above inequality with (2.4), we obtain that (2.5) holds also for $m > 1$.

Now, for $x \in \Omega(a_0)$, define

$$W(x) = \frac{1}{y(x)}(A\nabla y)(x).$$

Using (2.5) and $(W^T A^{-1} W)(x) \ge \lambda_{\max}^{-1}(x)|W(x)|^2$, we find that $W(x)$ satisfied the partial Riccati inequality

$$\begin{aligned} \operatorname{div} W(x) &\le -Q_1(x) - \langle W(x), b(x)A^{-1}(x)\rangle - (W^T A^{-1} W)(x) \\ &\le -Q_1(x) - \langle W(x), b(x)A^{-1}(x)\rangle - \lambda_{\max}^{-1}(x)|W(x)|^2. \end{aligned} \tag{2.6}$$

Multiplying (2.6) by $\rho(x)$, we get

$$\operatorname{div}(\rho(x)W(x)) \le -\rho(x)Q_1(x) + \langle W(x), \Phi(x)\rangle - \rho(x)\lambda_{\max}^{-1}(x)|W(x)|^2, \tag{2.7}$$

where $\Phi(x) = \nabla\rho(x) - \rho(x)b(x)A^{-1}(x)$. Multiplying (2.7) by $H_1(|x|, a_1)$ and integrating over the annulus $\Omega[a_1, c_1]$, we get

$$\begin{aligned} \int_{\Omega[a_1,c_1]} H_1(|x|, a_1)\rho(x)Q_1(x)\mathrm{d}x &\le -\int_{a_1}^{c_1} H_1(r, a_1)\int_{S(r)} \operatorname{div}(\rho(x)W(x))\mathrm{d}\sigma\mathrm{d}r \\ &\quad + \int_{\Omega[a_1,c_1]} H_1(|x|, a_1)\langle W(x), \Phi(x)\rangle\mathrm{d}x - \int_{\Omega[a_1,c_1]} H_1(|x|, a_1)\rho(x)\lambda_{\max}^{-1}(x)|W(x)|^2\mathrm{d}x. \end{aligned}$$

Integration by parts in the first integral of the right hand side of the above, and by Gauss formula, one can get

$$\begin{aligned} \int_{\Omega[a_1,c_1]} H_1(|x|, a_1)\rho(x)Q_1(x)\mathrm{d}x &\le -H_1(c_1, a_1)\int_{S(c_1)} \rho(x)\langle W(x), \nu(x)\rangle\mathrm{d}\sigma \\ &\quad + \int_{\Omega[a_1,c_1]} H_1(|x|, a_1)\rho(x)\langle W(x), \Theta_1(|x|, a_1)\rangle\mathrm{d}x - \int_{\Omega[a_1,c_1]} H_1(|x|, a_1)\rho(x)\lambda_{\max}^{-1}(x)|W(x)|^2\mathrm{d}x. \end{aligned} \tag{2.8}$$

Completing squares of $W(x)$ in (2.8) yields that

$$\int_{\Omega[a_1,c_1]} H_1(|x|,a_1)\rho(x)Q_1(x)\mathrm{d}x \le -H(c_1,a_1)\int_{S(c_1)} \rho(x)\langle W(x),\nu(x)\rangle \mathrm{d}\sigma$$
$$+\frac{1}{4}\int_{\Omega[a_1,c_1]} H_1(|x|,a_1)\rho(x)\lambda_{\max}(x)|\Theta_1(|x|,a_1)|^2\mathrm{d}x$$
$$-\int_{\Omega[a_1,c_1]} H_1(|x|,a_1)\rho(x)\lambda_{\max}^{-1}(x)\left|W(x)-\frac{1}{2}\lambda_{\max}(x)\Theta_1(|x|,a_1)\right|^2\mathrm{d}x.$$

Note that the third term of the right hand side is nonnegative, so,

$$\frac{1}{H_1(c_1,a_1)}\int_{\Omega[a_1,c_1]} H_1(|x|,a_1)\rho(x)\left[Q_1(x)-\frac{1}{4}\lambda_{\max}(x)|\Theta_1(|x|,a_1)|^2\right]\mathrm{d}x$$
$$\le -\int_{S(c_1)} \rho(x)\langle W(x),\nu(x)\rangle \mathrm{d}\sigma. \tag{2.9}$$

On the other hand, we begin with (2.7). Multiplying (2.7) by $H(b_1,|x|)$ and integrating over the annulus $\Omega[c_1,b_1]$, then proceeding as the above proof, we can obtain

$$\frac{1}{H_2(c_1,b_1)}\int_{\Omega[c_1,b_1]} H_2(b_1,|x|)\rho(x)\left[Q_1(x)-\frac{1}{4}\lambda_{\max}(x)|\Theta_2(b_1,|x|)|^2\right]\mathrm{d}x$$
$$\le \int_{S(c_1)} \phi(x)\langle W(x),\nu(x)\rangle \mathrm{d}\sigma. \tag{2.10}$$

Finally, adding (2.9) and (2.10), we get an inequality which contradicts the condition (2.2). Thus, we complete the proof. □

According to Theorem 2.1, it is easy for us to obtain the following immediate consequence.

Theorem 2.2. *Assume that for any $r \ge r_0$, there exist a_1, b_1, a_2, b_2 such that $r \le a_1 < b_1 \le a_2 < b_2$ and (2.1) holds. If there exist $c_i \in (a_i, b_i)$, $i = 1, 2$, $(H_1, H_2) \in \mathcal{H}$ and $\rho \in \mathbf{C}^1(\Omega[a_1,b_1] \cup \Omega[a_2,b_2], \mathbb{R}^+)$ such that*

$$\int_{\Omega[a_i,c_i]} H_1(|x|,a_i)\rho(x)\left[Q_1(x)-\frac{1}{4}\lambda_{\max}(x)|\Theta_1(|x|,a_i)|^2\right]\mathrm{d}x > 0, \tag{2.11}$$

and

$$\int_{\Omega[b_i,c_i]} H_2(b_i,|x|)\rho(x)\left[Q_1(x)-\frac{1}{4}\lambda_{\max}(x)|\Theta_1(b_i,|x|)|^2\right]\mathrm{d}x > 0, \tag{2.12}$$

for $i = 1, 2$, where $Q_1(x)$ is defined in Theorem 2.1, then Eq. (1.1) is oscillatory.

In case that $q_j(t)$ is nonpositive for $j = k+1, \ldots, m$, the following results provides some essentially new oscillation theorems for Eq. (1.1).

Theorem 2.3. *Assume that for any $r \ge r_0$, there exist a_1, b_1, a_2, b_2 such that $r \le a_1 < b_1 \le a_2 < b_2$ and*

$$\begin{cases} q_i(x) \ge 0 & x \in \Omega[a_1,b_1] \cup \Omega[a_2,b_2],\ i = 1,\ldots,k, \\ e(x) < 0, & x \in \Omega(a_1,b_1); \quad e(x) > 0,\ x \in \Omega(a_2,b_2). \end{cases} \tag{2.13}$$

If there exist $c_i \in (a_i,b_i)$, $i = 1,2$, $(H_1,H_2) \in \mathcal{H}$, $\rho \in \mathbf{C}^1(\Omega[a_1,b_1] \cup \Omega[a_2,b_2], \mathbb{R}^+)$ and positive numbers δ_i, $i = 1,\ldots,k$, and ε_j, $j = k+1,\ldots,m$, with $\sum_{i=1}^k \delta_i + \sum_{j=k+1}^m \varepsilon_j = 1$ such that

$$\frac{1}{H_1(c_i,a_i)}\int_{\Omega(a_i,c_i)} H_1(|x|,a_i)\rho(x)\left[Q_2(x)-\frac{1}{4}\lambda_{\max}(x)|\Theta_1(|x|,a_i)|^2\right]\mathrm{d}x$$
$$+\frac{1}{H_2(b_i,c_i)}\int_{\Omega(b_i,c_i)} H_2(b_i,|x|)\rho(x)\left[Q_2(x)-\frac{1}{4}\lambda_{\max}(x)|\Theta_2(b_i,|x|)|^2\right]\mathrm{d}x > 0, \tag{2.14}$$

for $i = 1, 2$, *where*

$$Q_2(x) = q(x) + \sum_{i=1}^{k} p_i |e(x)|^{1-1/\alpha_i} q_i^{1/\alpha_i}(x) - \sum_{j=k+1}^{m} p_j |e(x)|^{1-/\alpha_j} \bar{q}_j^{1/\alpha_j}(x),$$

with

$$p_i = \alpha_i(\alpha_i - 1)^{1/\alpha_i - 1} \delta_i^{1-1/\alpha_i}, \quad i = 1, \ldots, k,$$
$$p_j = \alpha_j(1 - \alpha_j)^{1/\alpha_j - 1} \varepsilon_j^{1-1/\alpha_j}, \quad j = k+1, \ldots, m,$$

and

$$\bar{q}_j(x) = \max\{-q_j(x), 0\}, \quad j = k+1, \ldots, m,$$

then Eq. (1.1) *is oscillatory.*

Proof. Assume to the contrary that $y = y(x)$ is a nonoscillatory solution and it is eventually positive say $y(x) > 0$ for all $x \in \Omega(a_0)$. In view of (A3), (1.1) can be rewritten as

$$\operatorname{div}(A(x)\nabla y) + \langle b(x), \nabla y\rangle + q(x)y + \sum_{i=1}^{k}[q_i(x)y^{\alpha_i} - \delta_i e(x)] + \sum_{j=k+1}^{m}[q_j(x)y^{\alpha_j} - \varepsilon_j e(x)] \le 0. \tag{2.15}$$

By Lemma 1.3, for $x \in \Omega(a_1, b_1)$, we get

$$\begin{aligned} q_i(x)y^{\alpha_i} - \delta_i e(x) &= q_i(x)y^{\alpha_i} + \delta_i |e(x)| \\ &\ge p_i q_i^{1/\alpha_i}(x)|e(x)|^{1-1/\alpha_i} y, \quad i = 1, \ldots, k, \end{aligned}$$

and

$$\begin{aligned} q_j(x)y^{\alpha_j} - \varepsilon_j e(x) &\ge -\left[\bar{q}_j(x)y^{\alpha_j} - \varepsilon_j |e(x)|\right] \\ &\ge -p_j \bar{q}_j^{1/\alpha_j}(x)|e(x)|^{1-1/\alpha_j} y, \quad j = k+1, \ldots, m. \end{aligned}$$

Therefore, we can obtain from (2.15) that

$$\operatorname{div}(A(x)\nabla y) + \langle b(x), \nabla y\rangle + Q_2(x)y \le 0 \tag{2.16}$$

which is the same as (2.5) in the proof of Theorem 2.1. The remaining argument is the same as that in Theorem 2.1. Thus this completes the proof of Theorem 2.3. □

By Theorem 2.3, we can obtain

Theorem 2.4. *Assume that for any* $r \ge r_0$, *there exist* a_1, b_1, a_2, b_2 *such that* $r \le a_1 < b_1 \le a_2 < b_2$ *and* (2.13) *holds. If there exist* $c_i \in (a_i, b_i)$, $i = 1, 2$, $(H_1, H_2) \in \mathcal{H}$, $\rho \in \mathbf{C}^1(\Omega[a_1, b_1] \cup \Omega[a_2, b_2], \mathbb{R}^+)$ *and positive numbers* δ_i, $i = 1, \ldots, k$, *and* ε_j, $j = k+1, \ldots, m$, *with* $\sum_{i=1}^{k}\delta_i + \sum_{j=k+1}^{m}\varepsilon_j = 1$, *such that*

$$\int_{\Omega(a_i,c_i)} H_1(|x|, a_i)\rho(x)\left[Q_2(x) - \frac{1}{4}\lambda_{\max}(x)|\Theta_1(|x|, a_i)|^2\right] dx > 0, \tag{2.17}$$

and

$$\int_{\Omega(b_i,c_i)} H_2(b_i, |x|)\rho(x)\left[Q_2(x) - \frac{1}{4}\lambda_{\max}(x)|\Theta_2(b_i, |x|)|^2\right] dx > 0, \tag{2.18}$$

for $i = 1, 2$, *where* $Q_2(x)$ *is defined in Theorem* 2.3, *then Eq.* (1.1) *is oscillatory.*

Remark 2.1. Let $N \equiv 1$, $b_i(x) \equiv 0$, $q_i(x) \equiv 0$ for all i, $e(x) \equiv 0$ in Eq. (1.1). Then Theorem 2.1 with $\rho(x) \equiv 1$ reduces Theorem 2.1 in Kong [18]. Furthermore, let $N = 1$ in Eq. (1.1), Theorems 2.1 and 2.4 with $\rho(x) \equiv 1$ reduce to Theorems 1 and 3 in Sun and Meng [19]. Thus our Theorems 2.1 and 2.3 are the generalizations of the main results in Sun and Meng [19].

Remark 2.2. For Eq. (1.3) with $N = 1$, Theorem 2.1 improves Theorem 2 in Yang [21].

Remark 2.3. Let $m \equiv 1$, $b_i(x) \equiv 0$, $q(x) \equiv 0$, and $\rho(x) \equiv 1$ in Theorem 2.1. Note that $\alpha_1(\alpha_1 - 1)^{1/\alpha_1 - 1} > 1$ for $\alpha_1 > 1$, so condition (2.2) in Theorem 2.1 is more general than the condition of Theorem 3 in Zhuang [15]. Therefore, Theorem 2.1 improves and extends the main results in Zhuang [15]. On the other hand, the method used in Zhuang [15], starting with integration of the Riccati equality over spheres in $\mathbb{R}^N$ centered in the origin, converts the $N-$dimensional problem into a problem in one variable and then extends Yang's results [21] to Eq. (1.3). In this paper, our methodology is somewhat different from Zhuang [15]. We believe that our approach is more natural for partial differential equations and provide much a deeper insight into oscillation.

Remark 2.4. Note that the test functions H_1 and H_2 in this paper may be different. However, as far as the author knows, all relevant results in literature suppose $H_1(r, s) = H_2(r, s)$, i.e., the same test function is used in the obtained oscillation criteria. Hence, the results established in this paper, by choosing functions $H_1(r, s) \neq H_2(r, s)$, are new in some sense even for the corresponding second order ordinary differential equations.

Remark 2.5. When there exist functions $c \in \mathbf{C}^{\nu}_{\text{loc}}(\Omega(r_0), \mathbb{R})$ and $f \in \mathbf{C}(\mathbb{R}, \mathbb{R}) \cap \mathbf{C}^1(\mathbb{R} - \{0\}, \mathbb{R})$ with $yf(y) > 0$ and $f'(y) \geq k > 0$ for $y \neq 0$ such that

$$C(x, y) \geq c(x) f(y) \quad \text{for all } x \in \Omega(r_0) \text{ and } y > 0, \tag{2.19}$$

the authors in papers [2,3,5–14] have established some oscillation criteria for Eq. (1.2). However, condition (A3) in present paper does not satisfy (2.19). Moreover, the results obtained here deal with cases in which we do not impose a restriction on the sign of $e(x)$ and $C(x, y)$ has the property of mixed nonlinearities.

3. Corollaries and examples

We now present a method of constructing test functions $H_1(r, s)$ and $H_2(r, s)$ as defined in Theorems 2.1–2.4. For illustration, we consider the simple case $m = 2$, hence $\alpha_1 > 1 > \alpha_2 > 0$.

Corollary 3.1. *Let $\alpha_1 > 1 > \alpha_2 > 0$, and let η_0, η_1, η_2 be positive constants with*

$$\eta_0 + \eta_1 + \eta_2 = 1, \qquad \alpha_1\eta_1 + \alpha_2\eta_2 = 1. \tag{3.1}$$

Assume that for any $r \geq r_0$, there exist a_1, b_1, a_2, b_2 such that $r \leq a_1 < b_1 \leq a_2 < b_2$ and

$$\begin{cases} q_i(x) \geq 0 & x \in \Omega[a_1, b_1] \cup \Omega[a_2, b_2],\ i = 1, 2, \\ e(x) \leq 0, & x \in \Omega[a_1, b_1]; \quad e(x) \geq 0,\ x \in \Omega[a_2, b_2]. \end{cases} \tag{3.2}$$

If there exist $c_i \in (a_i, b_i)$, $i = 1, 2$, $\rho \in \mathbf{C}^1(\Omega[a_1, b_1] \cup \Omega[a_2, b_2], \mathbb{R}^+)$ with $\nabla\rho(x) = \rho(x)b(x)A^{-1}(x)$, and $\alpha > 1$, $\beta > 1$ such that

$$\frac{1}{[\Lambda(c_i) - \Lambda(a_i)]^{\alpha-1}} \int_{\Omega[a_i,c_i]} [\Lambda(|x|) - \Lambda(a_i)]^{\alpha} \rho(x)\bar{Q}_1(x)\mathrm{d}x \geq \frac{\alpha^2}{4(\alpha - 1)}, \tag{3.3}$$

and

$$\frac{1}{[\Lambda(b_i) - \Lambda(c_i)]^{\beta-1}} \int_{\Omega[b_i,c_i]} [\Lambda(b_i) - \Lambda(|x|)]^{\beta} \rho(x)\bar{Q}_1(x)\mathrm{d}x \geq \frac{\beta^2}{4(\beta - 1)}, \tag{3.4}$$

for $i = 1, 2$, where

$$\bar{Q}_1(x) = q(x) + \left(\prod_{i=0}^{3} \eta_i^{-\eta_i}\right) |e(x)|^{\eta_0} q_1^{\eta_1}(x) q_2^{\eta_2}(x),$$

and

$$\lambda(r) = \int_{S(r)} \rho(x)\lambda_{\max}(x)\mathrm{d}\sigma, \qquad \Lambda(r) = \int_{r_0}^{r} \frac{\mathrm{d}s}{\lambda(s)},$$

then the equation

$$\operatorname{div}(A(x)\nabla y) + \langle b(x), \nabla y\rangle + q(x)y + q_1(x)|y|^{\alpha_1-1}y + q_2(x)|y|^{\alpha_2-1}y = e(x) \tag{3.5}$$

is oscillatory.

Proof. Let

$$H_1(r,s) = [\Lambda(r) - \Lambda(s)]^{\alpha}, \qquad H_2(r,s) = [\Lambda(r) - \Lambda(s)]^{\beta}.$$

Note that

$$h_1(r,s) = \frac{\alpha}{\lambda(r)[\Lambda(r) - \Lambda(s)]}, \qquad h_2(r,s) = -\frac{\beta}{\lambda(s)[\Lambda(r) - \Lambda(s)]},$$

and

$$\begin{aligned}\int_{\Omega[a_i,c_i]} H_1(|x|,a_i)\rho(x)\lambda_{\max}(x)|\Theta_1(|x|,a_i)|^2\mathrm{d}x &= \alpha^2\int_{a_i}^{c_i}[\Lambda(r) - \Lambda(a_i)]^{\alpha-2}\,\mathrm{d}\Lambda(r)\\ &= \frac{\alpha^2}{\alpha-1}[\Lambda(c_i) - \Lambda(a_i)]^{\alpha-1}\end{aligned} \tag{3.6}$$

from (3.3) and (3.6) we can get that (2.11) holds. Similarly, (3.4) implies that (2.12) holds. Hence, by Theorem 2.2, Eq. (3.5) is oscillatory. This completes the proof. □

Remark 3.1. η_0, η_1, η_2 in condition (3.1) of Corollary 3.1 can be chosen. For example, η_1, η_2 may be

$$\eta_1 = \frac{1 - \alpha_2(1-\eta_0)}{\alpha_1 - \alpha_2}, \qquad \eta_2 = \frac{\alpha_1(1-\eta_0) - 1}{\alpha_1 - \alpha_2},$$

where η_0 can be any positive number satisfying $0 < \eta_0 < (\alpha_1 - 1)/\alpha_1$. This will ensure that (3.1) holds.

Remark 3.2. For Eq. (1.3), let $\rho(x) \equiv 1$, Corollary 3.1 reduces Theorem 6 in Zhuang [15] by dropping the restricting condition $\lim_{r\to\infty}\Lambda(r) = \infty$.

Similar to the proof of Corollary 3.1, we have

Corollary 3.2. *Let $\alpha_1 > 1 > \alpha_2 > 0$. Assume that for any $r \geq r_0$, there exist a_1, b_1, a_2, b_2 such that $r \leq a_1 < b_1 \leq a_2 < b_2$ and*

$$\begin{cases} q_1(x) \geq 0 & x \in \Omega[a_1,b_1] \cup \Omega[a_2,b_2],\\ e(x) < 0, & x \in \Omega(a_1,b_1); \quad e(x) > 0,\ x \in \Omega(a_2,b_2).\end{cases} \tag{3.7}$$

If there exist $c_i \in (a_i, b_i)$, $i = 1, 2$, $\rho \in \mathbf{C}^1(\Omega[a_1,b_1] \cup \Omega[a_2,b_2], \mathbb{R}^+)$ with $\nabla\rho(x) = \rho(x)b(x)A^{-1}(x)$, and $\alpha > 1$, $\beta > 1$ such that

$$\frac{1}{[\Lambda(c_i) - \Lambda(a_i)]^{\alpha-1}}\int_{\Omega(a_i,c_i)}[\Lambda(|x|) - \Lambda(a_i)]^{\alpha}\,\rho(x)\bar{Q}_2(x)\mathrm{d}x \geq \frac{\alpha^2}{4(\alpha-1)}, \tag{3.8}$$

and

$$\frac{1}{[\Lambda(b_i) - \Lambda(c_i)]^{\beta-1}}\int_{\Omega(c_i,b_i)}[\Lambda(b_i) - \Lambda(|x|)]^{\beta}\,\rho(x)\bar{Q}_2(x)\mathrm{d}x \geq \frac{\beta^2}{4(\beta-1)}, \tag{3.9}$$

for $i = 1, 2$, where

$$\begin{aligned}\bar{Q}_2(x) &= q(x) + \alpha_1(\alpha_1 - 1)^{1/\alpha_1 - 1}\delta^{1-1/\alpha_1}|e(x)|^{1-1/\alpha_1}q_1^{1/\alpha_1}(x)\\ &\quad - \alpha_2(1-\alpha_2)^{1/\alpha_2 - 1}(1-\delta)^{1-1/\alpha_2}|e(x)|^{1-1/\alpha_2}\bar{q}_2^{\,1/\alpha_2}(x),\end{aligned}$$

$0 < \delta < 1$, $\bar{q}_2(x) = \max\{-q_2(x), 0\}$, *and $\Lambda(r)$ is defined in Corollary 3.1, then Eq. (3.5) is oscillatory.*

Finally, we give two examples to illustrate our main results.

Example 3.1. Consider Eq. (3.5) with

$$N=2,\qquad A=I \text{ (identity matrix)},\qquad b(x)=\left(-\frac{x_1}{|x|^2},-\frac{x_2}{|x|^2}\right),$$
$$q(x)=\gamma\sin|x|,\qquad q_1(x)=\gamma_1,\qquad q_2(x)=\gamma_2,\qquad e(x)=\sin^3|x|, \tag{3.10}$$

where $x\in\Omega(1)$, $\alpha_1=5/2$, $\alpha_2=1/2$, and γ,γ_1,γ_2 are positive constants. For any $r\geq 1$, let $a_1=(2k-1)\pi$, $b_1=a_2=2k\pi$ and $b_2=(2k+1)\pi$, where $k=1,2,\ldots$. We choose $\eta_0=\eta_1=\eta_2=1/3$ so that (3.1) holds. It is easy to see that the sign condition (3.2) in Corollary 3.1 is satisfied.

Let $\rho(x)=1/|x|$. A direct computation yields that

$$\rho(x)b(x)A^{-1}(x)=\nabla\rho(x),\qquad \lambda(r)=2\pi,\qquad \bar{Q}_1(x)=\left[\gamma+3(\gamma_1\gamma_2)^{1/3}\right]|\sin|x||.$$

Now, setting $\alpha=\beta=2$, $c_1=(2k-1)\pi+\pi/2$ and $c_2=2k\pi+\pi/2$ in Corollary 3.1, we find that

$$\begin{aligned}\int_{\Omega[a_i,c_i]}[\Lambda(|x|)-\Lambda(a_i)]^\alpha\,\rho(x)\bar{Q}_1(x)\mathrm{d}x &= \frac{\gamma+3(\gamma_1\gamma_2)^{1/3}}{2\pi}\int_0^{\pi/2}s^2\sin s\,\mathrm{d}s\\ &=\frac{\pi-2}{2\pi}\left[\gamma+3(\gamma_1\gamma_2)^{1/3}\right],\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega[c_i,b_i]}[\Lambda(b_i)-\Lambda(|x|)]^\beta\,\rho(x)\bar{Q}_1(x)\mathrm{d}x &= \frac{\gamma+3(\gamma_1\gamma_2)^{1/3}}{2\pi}\int_0^{\pi/2}s^2\sin s\,\mathrm{d}s\\ &=\frac{\pi-2}{2\pi}\left[\gamma+3(\gamma_1\gamma_2)^{1/3}\right].\end{aligned}$$

Therefore, by Corollary 3.1, if we choose that γ,γ_1,γ_2 such that

$$\gamma+3(\gamma_1\gamma_2)^{1/3}>\frac{\pi}{2(\pi-2)},$$

which implies that (3.3) and (3.4) hold, then Eq. (3.10) is oscillatory.

Example 3.2. Consider Eq. (3.5) with

$$N=2,\qquad A=\operatorname{diag}\left(\frac{1}{|x|},\frac{1}{|x|}\right),\qquad b(x)=(0,0),\qquad q(x)=\frac{\gamma}{|x|}\sin|x|,$$
$$q_1(x)=\frac{\gamma_1}{|x|^3},\qquad q_2(x)=\gamma_2\cos|x|,\qquad e(x)=-|x|\cos|x|, \tag{3.11}$$

where $x\in\Omega(1)$, $\alpha_1=2$, $\alpha_2=1/2$, and γ,γ_1,γ_2 are positive constants. For any $r\geq 1$, let $a_1=2k\pi-\pi/2$, $b_1=a_2=2k\pi+\pi/2$ and $b_2=2k\pi+3\pi/2$, where $k=1,2,\ldots$. It is easy to see that the sign condition (3.7) in Corollary 3.2 is satisfied.

Let $\rho(x)=1$ and $\delta=1/2$. A direct computation yields that

$$\rho(x)b(x)A^{-1}(x)=\nabla\rho(x),\qquad \lambda(r)=2\pi,$$

and

$$\bar{Q}_2(x)=\begin{cases}\dfrac{1}{|x|}\left[c\gamma\sin|x|+\sqrt{2c\gamma_1}\cos^{1/2}|x|\right], & x\in\Omega(a_1,b_1),\\ \dfrac{1}{|x|}\left[\gamma\sin|x|+\sqrt{2\gamma_1}|\cos|x||^{1/2}-\dfrac{1}{2}\gamma_2^2|\cos|x||\right], & x\in\Omega(a_2,b_2).\end{cases}$$

For Corollary 3.2, let $\alpha=\beta=2$, $c_1=2k\pi$ and $c_2=(2k+1)\pi$. It is easy to see that

$$\int_{\Omega[a_1,c_1]} [\Lambda(|x|) - \Lambda(a_1)]^\alpha \rho(x) \bar{Q}_2(x) \mathrm{d}x = \frac{1}{2\pi} \int_{-\pi/2}^{0} \left(r + \frac{\pi}{2}\right)^2 \left[\gamma \sin r + \sqrt{2c\gamma_1} \cos^{1/2} r\right] \mathrm{d}r$$
$$= \frac{1}{2\pi} \left[\gamma \left(2 - \frac{\pi^2}{4}\right) + \sqrt{2\gamma_1} B \left(2, \frac{1}{2}\right)\right],$$

and

$$\int_{\Omega[c_1,b_1]} [\Lambda(b_1) - \Lambda(|x|)]^\beta \rho(x) \bar{Q}_2(x) \mathrm{d}x = \frac{1}{2\pi} \int_{0}^{\pi/2} \left(\frac{\pi}{2} - r\right)^2 \left[\gamma \sin r + \sqrt{2\gamma_1} \cos^{1/2} r\right] \mathrm{d}r$$
$$= \frac{1}{2\pi} \left[-\gamma \left(2 - \frac{\pi^2}{4}\right) + \sqrt{2\gamma_1} B \left(2, \frac{1}{2}\right)\right].$$

Similarly, we have

$$\int_{\Omega[a_2,c_2]} [\Lambda(|x|) - \Lambda(a_2)]^\alpha \rho(x) \bar{Q}_2(x) \mathrm{d}x = \frac{1}{2\pi} \left[(2\gamma - \gamma_2^2) \left(\frac{\pi^2}{8} - 1\right) + \sqrt{2\gamma_1} B \left(2, \frac{1}{2}\right)\right],$$

and

$$\int_{\Omega[c_2,b_2]} [\Lambda(b_2) - \Lambda(|x|)]^\beta \rho(x) \bar{Q}_2(x) \mathrm{d}x = \frac{1}{2\pi} \left[-(2\gamma - \gamma_2^2) \left(\frac{\pi^2}{8} - 1\right) + \sqrt{2\gamma_1} B \left(2, \frac{1}{2}\right)\right],$$

where $B(2, 1/2) = \int_0^{\pi/2} s^2 \sin^{1/2} s \mathrm{d}s$. Therefore, by Corollary 3.2, if we choose that $\gamma, \gamma_1, \gamma_2$ such that

$$2\gamma \geq \gamma_2^2 \quad \text{and} \quad \sqrt{2\gamma_1} B \left(2, \frac{1}{2}\right) > \frac{\pi}{2} + \frac{1}{4}\gamma(\pi^2 - 8),$$

which implies that (3.8) and (3.9) hold, then Eq. (3.11) is oscillatory.

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